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Generation of Irregular Disturbances and Solitarylike Waves in Transitional Boundary Layers

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The interrelation between erratic signals and large-sized coherent structures excited in boundary layers transitioning to turbulence is investigated. To shed light on the issue of fundamental importance, a new dynamical system has been put forward recently on the basis of simple physical arguments assuming the Reynolds number to tend to infinity. Wave dispersion in the system is defined by the Laplacian of the self-induced pressure resulting from inviscid–inviscid interaction of the intermediate (adjustment) sublayer with the outer oncoming stream. This regime supersedes the triple-deck velocity pattern when the pulsation size attains sufficiently large magnitudes. In the present paper, asymptotic equations governing a three-dimensional disturbance field are expanded into a power series of a small parameter proportional to the magnitude of excess pressure. However, the final closed-form result does not depend on this parameter. The birth of erratic signals typical of transitional boundary-layer flow is demonstrated by computing the two-dimensional disturbance field downstream of a steady obstacle. In accordance with experimental findings, solitarylike coherent structures emerge amid the foam of erratic oscillations surviving the random buffeting by the surrounding fluid. The phase speed of solitarylike waves exceeds the speed of weaker pulsations; therefore, the large-sized waves leave behind the small-sized signals.

Nomenclature

A	=	instantaneous displacement thickness
B	=	constant
F	=	auxiliary function
k	=	streamwise wave number
$\mathcal{L}$	=	interaction law operator
m	=	spanwise wave number
p	=	self-induced pressure
Q	=	auxiliary function
Re	=	Reynolds number
t	=	time
u, v, w	=	velocity components
x, y, z	=	Cartesian coordinates
$\Delta, \delta, \varepsilon$	=	small parameters
τ	=	skin friction
ω	=	frequency

Subscripts

w	=	wall condition
1, 2, 3	=	term order in asymptotic expansions
$\bar{}$	=	amplitude functions

I. Introduction

TO DEAL with large-sized coherent structures intrinsic to a boundary layer transitioning to turbulence, a pertinent theoretical model should be worked out starting from essentially nonlinear phases of the stability loss. However, the germ of the idea dates back to the distant past, and the history of the experimental as well as theoretical attempts in hydrodynamics to provide an adequate description of nonlinear solitary disturbances begins with Russel [1] reporting on his observations of the large "wave of translation" in the Union canal near Edinburgh, Scotland. Some 60 years later, Korteweg and de Vries [2] derived a third-order partial differential equation, currently referred to by their names, to explain that "rare and beautiful phenomenon" occurring in shallow-water channels. After 70 more years, the Korteweg–de Vries (KDV) equation was explicitly solved by Gardner et al. [3,4], and their solution gave rise to an entire new mathematical field, the soliton theory (see Ablowitz and Segur [5] and Ablowitz and Clarkson [6]). At the same time, Benjamin [7] and Davis and Acrivos [8] independently arrived at an integral partial differential equation governing internal waves in deep water. The difference in the KDV and Benjamin–Davis–Acrivos (BDA) dispersion terms stems from the fact that the former relates to long-wavelength fluid motions, whereas short-scaled pulsations obey the latter.

When processing experimental data from their wind-tunnel tests, Borodulin and Kachanov [9,10] made an important observation that solitonlike coherent structures are an integral part of transitional flows, without identifying their nature. The direct relevance of BDA solitons to experimentally recorded large coherent formations was established by Kachanov et al. [11] on the basis of an earlier work by Zhuk and Ryzhov [12] and Smith and Burggraf [13]. This study appears to be the first to bridge the theory of solitons and a high-Reynolds-number approach to large-sized disturbances in the boundary layer transitioning to turbulence.

More recently, an evolution equation was introduced in hydrodynamics that incorporates both BDA and KDV dispersion terms; it covers a broad scope of spatial scales. The compound BDA–KDV equation came from the work of Benjamin [14] on solitary waves, which can occur in a two-fluid system in which a thin upper layer of incompressible fluid lies on a deep, more dense, also incompressible fluid. The interface separating the two fluids is subject to capillarity. Independently, a similar equation was put into operation as applied to completely different environmental conditions and spectral range, in an invited lecture delivered by Ryzhov [15] at the First European Fluid Mechanics Conference. The focus in this lecture was on the nonlinear stability of the boundary layer. In a subsequent paper, Benjamin [16] published an in-depth mathematical treatment of the compound BDA–KDV equation using positive-operator methods. The birth of large-sized solitary waves amid erratically propagating lower-amplitude signals emitted by a steady hump was demonstrated by Ryzhov and Bogdanova-Ryzhova [17].

A limitation of all the studies cited previously lies in the premise that solitary waves depend on time and the only spatial coordinate. As a result, dynamical systems cannot encompass many important

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properties of real events. To overcome this restriction, Kadomtsev and Petviashvili [18] advanced, on the basis of very simple physical reasoning, a weakly two-dimensional extension of the KDV equation in which nonlinearity, dispersion, and two-dimensionality are of equal order in magnitude. Some important properties of the Kadomtsev–Petviashvili (KP) equation are mentioned in Drazin and Johnson [19]; Ablowitz and Clarkson [6] present its thorough analysis. At present, the KP equation finds many applications in describing solitary as well as nonlinear periodic waves in various branches of physics and astronomy (see Petviashvili and Pokhotelov [20]).

No two-dimensional generalization of the BDA or compound BDA–KDV dynamical systems was put forward until very recently. Starting from a clear physical consideration, Ryzhov [21,22] devised an approach to developing the dispersion characteristics of short-scaled solitary waves in shear flows of the boundary-layer type. The self-induced pressure resulting from inviscid–inviscid interaction sustains a mechanism that controls the disturbance propagation, regardless of the interaction law, be it of the KDV differential, BDA integral differential, or compound BDA–KDV form. When the dependence on the second spatial coordinate becomes negligible, the new evolution equations converge to their respective one-dimensional limits. However, the new two-dimensional KDV-type equation does not coincide with the KP equation, though both of them tend to the same one-dimensional limit. The reason is that they relate to different frequency and wave-number spectra of disturbances. Then, Ryzhov [23] directed the way to provide the necessary background for the physics-based arguments. This effort is completed in the present paper, by expanding a nonlinear solution into a series in powers of a small parameter and explicitly determining a function to which the series converges. A similar technique allows the boundary condition on the solid surface to be cast in a simple form. The dynamical system, an outcome of the nonlinear analysis, involves a quadratic term responsible for the disturbance generation. The birth of large-sized solitarylike coherent structures amid and from lower-amplitude erratic signals is demonstrated within the framework of a one-dimensional version of the compound BDA–KDV evolution equation. Notwithstanding a limited nature of the simplified version, the computation shows that the large-sized disturbances survive the random buffeting by the surrounding fluid and continue to build up over long distances. Thus, the new dynamical system is capable of predicting both the irregular pulsations and well-organized motions, which come hand in hand. According to a thorough survey of experimental data by Cantwell [24] and Kachanov [25], these silent features are intrinsic to transitional and fully developed turbulent boundary layers.

II. Adjustment Sublayer

Here, we put forth a set of equations together with the limit and boundary conditions that control solitary as well as nonlinear periodic waves in incompressible boundary-layer flows. These short-scaled, large-sized disturbances develop in the so-called adjustment sublayer, introduced by Zhuk and Ryzhov [12] and Smith and Burggraf [13] in their stability studies on the assumption that the Reynolds number $Re \to \infty$. The inviscid adjustment sublayer is sandwiched between the main body of the boundary layer and the near-wall viscous sublayer. The asymptotic description of the adjustment sublayer requires two small parameters, $\varepsilon = Re^{-1/8}$ and Δ, such that

$$\epsilon \ll \Delta \ll 1 \tag{1}$$

The first is typical of the triple-deck theory (see Stewartson [26] and Smith [27]). The second parameter serves to calibrate the amplitude of the self-induced pressure when it attains the magnitude of order Δ^2.

Assuming the wall skin friction τ_w in the initially undisturbed state to not be vanishingly small, let us introduce the following scaled nondimensional variables:

$$\bar{t} = \epsilon^4 \Delta^{-2} \tau_w^{-3/2} t \tag{2}$$

$$\bar{x} = 1 + \epsilon^4 \Delta^{-1} \tau_w^{-5/4} x \tag{3}$$

$$\bar{y} = \epsilon^4 \Delta \tau_w^{-3/4} y \tag{4}$$

$$\bar{z} = \epsilon^4 \Delta^{-1} \tau_w^{-5/4} z \tag{5}$$

and define in the adjustment sublayer the incompressible velocity field and the self-induced pressure by

$$\bar{u} = \Delta(\tau_w^{1/4} u^{(1)} + \Delta u^{(2)} + \epsilon^4 \Delta^{-4} u^{(3)} + \ldots) \tag{6}$$

$$\bar{v} = \Delta^3(\tau_w^{3/4} v^{(1)} + \Delta v^{(2)} + \epsilon^4 \Delta^{-4} v^{(3)} + \ldots) \tag{7}$$

$$\bar{w} = \Delta(\tau_w^{1/4} w^{(1)} + \Delta w^{(2)} + \epsilon^4 \Delta^{-4} w^{(3)} + \ldots) \tag{8}$$

$$\bar{p} = \Delta^2(\tau_w^{1/2} p^{(1)} + \Delta p^{(2)} + \epsilon^4 \Delta^{-4} p^{(3)} + \ldots) \tag{9}$$

According to Eq. (4), the adjustment sublayer is thinner than the boundary layer where the Prandtl normal-to-wall distance $\bar{y} = \varepsilon^4 y_2$. Notice that the third-order terms in the asymptotic expansions (6–9) are smaller than the second-order terms, provided that $\varepsilon^{4/5} \ll \Delta$, but in the range $\varepsilon \ll \Delta \ll \varepsilon^{4/5}$ their positions should be interchanged. As for the leading-order terms, they retain their positions regardless of the relative magnitude of Δ in Eq. (1). Only the leading-order terms are treated next, and for this reason the superscript *1* will be subsequently omitted when labelling the desired leading-order functions.

Substitution of Eqs. (6–9) into the full system of Navier–Stokes equations, scaled according to Eqs. (2–5), yields a set of boundary-layer-type equations:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{10}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{\partial p}{\partial x} \tag{11}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z} \tag{12}$$

where the viscous stresses are neglected. The self-induced pressure $p = p(t, x, z)$ has to be evaluated simultaneously with the three-dimensional velocity field. The interaction law

$$p = \mathcal{L}(A) \tag{13}$$

to relate the self-induced pressure to the instantaneous displacement thickness $-A(t, x, z)$ comes from matching of the solutions for all sublayers. A pertinent analysis for the two-dimensional setting initially performed by Zhuk and Ryzhov [12] and Smith and Burggraf [13] was extended to the general three-dimensional case in Kachanov et al. [11].

No matter the operator $\mathcal{L}$ in Eq. (13), the limit conditions, as $y \to \infty$, derived from matching a solution for the adjustment sublayer with solutions for the sublayers on top, read

$$u \to y + A(t, x, z) + \frac{1}{y}\int_{-\infty}^{x} \mathrm{d}\xi_1 \int_{-\infty}^{\xi_1} \frac{\partial^2 p(t, \xi_2, z)}{\partial z^2}\, \mathrm{d}\xi_2 \tag{14}$$

$$v \to -\frac{\partial A}{\partial x} y - \frac{\partial A}{\partial t} - A\frac{\partial A}{\partial x} - \frac{\partial p}{\partial x} - \int_{-\infty}^{x} \frac{\partial^2 p(t, \xi_1, z)}{\partial z^2}\, \mathrm{d}\xi_1 \tag{15}$$

$$w \to -\frac{1}{y}\int_{-\infty}^{x} \frac{\partial p(t, \xi_1, z)}{\partial z}\, \mathrm{d}\xi_1 \tag{16}$$

The boundary condition

$$v = \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + w\frac{\partial h}{\partial z} \quad \text{at } y = h(t, x, z), \qquad t \geq 0 \tag{17}$$

with h being equal to zero for $t < 0$, depends on the shape of a vibrator (or steady obstacle) on an otherwise smooth surface. If $h = 0$ for all $-\infty < t < \infty$, we are led to solve a nonlinear problem in eigenvalues that defines solitary and periodic disturbances.

There exists an important consequence, ensuing from the set of governing equations. To derive it, we differentiate Eq. (11) with respect to x:

$$\frac{\partial^2 u}{\partial t \partial x} + u\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 p}{\partial x^2} = -\frac{\partial}{\partial x}\left(v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) - \left(\frac{\partial u}{\partial x}\right)^2$$

Differentiation of Eq. (12) with respect to z gives

$$\frac{\partial^2 w}{\partial t \partial z} + u\frac{\partial^2 w}{\partial z^2} + \frac{\partial^2 p}{\partial z^2} = -\frac{\partial}{\partial z}\left(v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) - \frac{\partial u}{\partial z}\frac{\partial w}{\partial x}$$

Combining the last two equations results in

$$\left(\frac{\partial}{\partial t} + y\frac{\partial}{\partial x}\right)F + \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)p = -\frac{\partial}{\partial x}\left(v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) - \frac{\partial}{\partial z}\left(v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) - \left(\frac{\partial u}{\partial x}\right)^2 - \frac{\partial u}{\partial z}\frac{\partial w}{\partial x} \tag{18}$$

where an auxiliary function

$$F = \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \tag{19}$$

is critical to the next analysis. Like the original set of governing equations, the differential consequence [Eq. (18)] satisfies the limit conditions (14–16), in which the interaction law can be chosen arbitrarily.

III. Power Series Expansions

An initial-equilibrium state of the adjustment sublayer is supposed to be specified by

$$u = y, \qquad v = w = p = A = 0 \tag{20}$$

Accordingly, the disturbance field evolves under the action of a perturbing agency [Eq. (17)] that is put into operation at some moment $t = 0$, when no departures from the initial equilibrium are present. Let us introduce a positive parameter:

$$0 < \delta < 1 \tag{21}$$

and write the expansions

$$u = y + u' = y + \delta u_1 + \delta^2 u_2 + \delta^3 u_3 + \dots + \delta^{n-1}u_{n-1} + \delta^n u_n + \dots \tag{22}$$

$$v = v' = \delta v_1 + \delta^2 v_2 + \delta^3 v_3 + \dots + \delta^{n-1}v_{n-1} + \delta^n v_n + \dots \tag{23}$$

$$w = w' = \delta w_1 + \delta^2 w_2 + \delta^3 w_3 + \dots + \delta^{n-1}w_{n-1} + \delta^n w_n + \dots \tag{24}$$

$$F = F' = \delta F_1 + \delta^2 F_2 + \delta^3 F_3 + \dots + \delta^{n-1}F_{n-1} + \delta^n F_n + \dots \tag{25}$$

$$A = A' = \delta A_1 + \delta^2 A_2 + \delta^3 A_3 + \dots + \delta^{n-1}A_{n-1} + \delta^n A_n + \dots \tag{26}$$

$$p = p' = \delta p_1 + \delta^2 p_2 + \delta^3 p_3 + \dots + \delta^{n-1}p_{n-1} + \delta^n p_n + \dots \tag{27}$$

for the velocity field, the displacement thickness, and the excess pressure that represent deviations from the state of equilibrium brought about by the forced oscillations.

For the sake of convenience, we recast Eq. (18) in a symmetric form:

$$\left(\frac{\partial}{\partial t} + y\frac{\partial}{\partial x}\right)F' + \frac{\partial v'}{\partial x} + \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)p' = -\left[\frac{\partial}{\partial x}\left(u'\frac{\partial u'}{\partial x} + v'\frac{\partial u'}{\partial y} + w'\frac{\partial u'}{\partial z}\right) + \frac{\partial}{\partial z}\left(u'\frac{\partial w'}{\partial x} + v'\frac{\partial w'}{\partial y} + w'\frac{\partial w'}{\partial z}\right)\right] \tag{28}$$

with the wall shear stress explicitly isolated:

$$F' = \frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = -\frac{\partial v'}{\partial y} \tag{29}$$

owing to Eqs. (10) and (19). Letting for the time being

$$\delta \to 0$$

and collecting in Eq. (28) terms of equal order in δ, we have the first three relations:

$$\left(\frac{\partial}{\partial t} + y\frac{\partial}{\partial x}\right)F_1 + \frac{\partial v_1}{\partial x} + \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)p_1 = Q_1 = 0 \tag{30a}$$

$$\left(\frac{\partial}{\partial t} + y\frac{\partial}{\partial x}\right)F_2 + \frac{\partial v_2}{\partial x} + \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)p_2 = Q_2 = -\left[\frac{\partial}{\partial x}\left(u_1\frac{\partial u_1}{\partial x} + v_1\frac{\partial u_1}{\partial y} + w_1\frac{\partial u_1}{\partial z}\right) + \frac{\partial}{\partial z}\left(u_1\frac{\partial w_1}{\partial x} + v_1\frac{\partial w_1}{\partial y} + w_1\frac{\partial w_1}{\partial z}\right)\right] \tag{30b}$$

$$\left(\frac{\partial}{\partial t} + y\frac{\partial}{\partial x}\right)F_3 + \frac{\partial v_3}{\partial x} + \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)p_3 = Q_3 = -\left[\frac{\partial}{\partial x}\left(\frac{\partial u_1 u_2}{\partial x} + v_2\frac{\partial u_1}{\partial y} + v_1\frac{\partial u_2}{\partial y} + w_2\frac{\partial u_1}{\partial z} + w_1\frac{\partial u_2}{\partial z}\right) + \frac{\partial}{\partial z}\left(u_2\frac{\partial w_1}{\partial x} + u_1\frac{\partial w_2}{\partial x} + v_2\frac{\partial w_1}{\partial y} + v_1\frac{\partial w_2}{\partial y} + \frac{\partial w_1 w_2}{\partial z}\right)\right] \tag{30c}$$

Notice that the left-hand sides of Eqs. (30a–30c) are similar in form. Differentiating Eq. (30a), with respect to y, leads to

$$\left(\frac{\partial}{\partial t} + y\frac{\partial}{\partial x}\right)\frac{\partial F_1}{\partial y} = 0 \tag{31}$$

from which a key property

$$F_1 = F_1(t, x, z) \tag{32}$$

of the auxiliary function F follows in the first approximation.

IV. Free Oscillations

Before proceeding to a rigorous examination of the general nonlinear case, it is desirable to treat in detail the first three approximations set out in Eqs. (30a–30c). This preliminary study is aimed at elucidating the essentials of the problem. A general consideration of higher-order approximations will be given in the next section.

A. Linear Analysis

The homogeneous Eq. (30a) was thoroughly investigated in Ryzhov [21,22] on the assumption that neither a steady nor a vibrating obstacle resides on a smooth surface at the bottom of a boundary layer. Then, $h = 0$ in the boundary condition (17), which reduces to

$$v = 0 \quad \text{at } y = 0 \tag{33}$$

With this constraint imposed, Eq. (30a) converges to

$$\frac{\partial F_1}{\partial t} + \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right) p_1 = 0 \tag{34}$$

Neither of the two functions F_1 and p_1 appearing here depend on y; therefore, Eq. (34) still holds at the upper edge $y \to \infty$ of the linear adjustment sublayer. It follows from the limit conditions (14) and (16), expanded in terms of δ, that

$$F_1 = \frac{\partial A_1}{\partial x} \tag{35}$$

and, hence,

$$\frac{\partial^2 A_1}{\partial t \partial x} + \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right) p_1 = 0 \tag{36}$$

The last relation, supplemented with the interaction law $p_1 = \mathcal{L}(A_1)$, controls the propagation of small-amplitude waves in the adjustment sublayer. The most important inference to be drawn from Eq. (36) is that dispersion of spatial disturbances obeys the Laplace operator of the self-induced pressure and, as a consequence, turns out to be homogeneous in both directions, x and z. We will be guided by this observation in the subsequent analysis.

The expression

$$v_1 = -yF_1 = -y\frac{\partial A_1}{\partial x} \tag{37}$$

for the transverse velocity derives from Eqs. (10) and (35). As it suggests, v_1 is a function of both velocity components u_1 and w_1 in a plane $y = \text{const}$ parallel to the solid surface. To obtain the other components of the velocity and pressure fields in terms of the instantaneous displacement thickness, let us introduce a solution:

$$(u_1, v_1, w_1, F_1, p_1, A_1) = [\bar{U}_1(y), \bar{V}_1(y), \bar{W}_1(y), \bar{F}_1, \bar{P}_1, \bar{A}_1] \times \exp[\omega t + i(kx + mz)] \tag{38}$$

of the travelling-wave type with

$$\bar{V}_1 = -iky\bar{A}_1 \tag{39}$$

directly ensuing from Eq. (37). As is easily seen, Eq. (36) yields the first part:

$$\bar{P}_1 = \frac{i\omega k}{k^2 + m^2}\bar{A}_1 \tag{40}$$

of the dispersion relation to link $\bar{P}_1$ and $\bar{A}_1$. As usual, the second part comes from the interaction law, not specified so far. The lateral velocity

$$\bar{W}_1 = -\frac{im}{\omega + iky}\bar{P}_1 = \frac{\omega k m}{(k^2 + m^2)(\omega + iky)}\bar{A}_1 \tag{41}$$

is found from an expansion of Eq. (12), in terms of δ. Then, Eq. (10) leads to the expression

$$\bar{U}_1 = \bar{A}_1 + \frac{im^2}{k(\omega + iky)}\bar{P}_1 \tag{42}$$

to determine the streamwise component of velocity. Summing up $ik\bar{U}_1$ and $im\bar{W}_1$ results in

$$\bar{F}_1 = i(k\bar{U}_1 + m\bar{W}_1) = ik\bar{A}_1 \tag{43}$$

in accord with Eqs. (29) and (35). The evolution Eq. (34) assumes the canonical form of Eq. (36).

B. Second-Order Approximation

The argument $2[\omega t + i(kx + mz)]$ of the exponential factor in the second-order travelling-wave-type solution

$$(u_2, v_2, w_2, F_2, p_2, A_2) = [\bar{U}_2(y), \bar{V}_2(y), \bar{W}_2(y), \bar{F}_2, \bar{P}_2, \bar{A}_2]\exp\{2[\omega t + i(kx + mz)]\} \tag{44}$$

is twice the argument of the exponential factor in Eq. (38). With the first-order velocity field determined previously, the right-hand side of Eq. (30b) can be evaluated directly, though the algebra is tedious. The final expression

$$Q_2 = \bar{Q}_2(y)\exp\{2[\omega t + i(kx + mz)]\} \tag{45}$$

with the amplitude factor

$$\bar{Q}_2 = 2k^2\bar{A}_1^2 \tag{46}$$

implies that

$$Q_2 = -\frac{\partial}{\partial x}\left(A_1\frac{\partial A_1}{\partial x}\right) \tag{47}$$

A shortcut to deriving Eqs. (45–47) is to put

$$\bar{F}_2 = 2i(k\bar{U}_2 + m\bar{W}_2) \tag{48}$$

based on Eq. (29), and to take advantage of the auxiliary functions $\bar{F}_1$ and $\bar{F}_2$ in the expression

$$(\omega + iky)\bar{F}_2 + ik\bar{V}_2 - 2(k^2 + m^2)\bar{P}_2 = \frac{1}{2}\bar{Q}_2 = -\left\{ik\left[\bar{U}_1(ik\bar{U}_1 + im\bar{W}_1) + \bar{V}_1\frac{d\bar{U}_1}{dy}\right] + im\left[\bar{W}_1(ik\bar{U}_1 + im\bar{W}_1) + \bar{V}_1\frac{d\bar{W}_1}{dy}\right]\right\} \tag{49}$$

which arises as the result of the substitution of Eqs. (44) and (48) into Eq. (30b). Simple manipulation reduces the right-hand side of Eq. (49) to

$$\frac{1}{2}\bar{Q}_2 = -\bar{F}_1^2 - \bar{V}_1\frac{d\bar{F}_1}{dy} = -\bar{F}_1^2 \tag{50}$$

insofar as $\bar{F}_1$ does not depend on y. Differentiating Eq. (49) with respect to y leaves us with

$$(\omega + iky)\frac{d\bar{F}_2}{dy} = 0 \tag{51}$$

in spite of the fact that $\bar{Q}_2$ does not equal zero. Thus, the key property

$$F_2 = F_2(t, x, z) \tag{52}$$

of the auxiliary function F holds in the second approximation. Because

$$F_2 = \frac{\partial A_2}{\partial x} \tag{53}$$

from the limit conditions (14) and (16) expanded in terms of δ, we can put

$$\bar{F}_2 = 2ik\bar{A}_2 \tag{54}$$

for any y. Taking into consideration the boundary condition $\bar{V}_2 = 0$ at $y = 0$, Eq. (49) becomes

$$2i\omega k\bar{A}_2 - 2(k^2 + m^2)\bar{P}_2 = k^2\bar{A}_1^2 \tag{55}$$

with $ik\bar{A}_1$ and $2ik\bar{A}_2$ substituted for $\bar{F}_1$ and $\bar{F}_2$, respectively. Here, none of the three functions $\bar{A}_1$, $\bar{A}_2$, and $\bar{P}_2$, depends on y. The term

$k^2\bar{A}_1^2$ is $1/2\bar{Q}_2$, on the strength of Eq. (50). With allowance made for the expression

$$\frac{\partial^2 A_2}{\partial t \partial x} = 4i\omega k \bar{A}_2 e^{2[\omega t + i(kx+mz)]}$$

we infer, from Eq. (55), that the second-order evolution equation reads

$$\frac{\partial^2 A_2}{\partial t \partial x} + \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right) p_2 = -\frac{\partial}{\partial x}\left(A_1 \frac{\partial A_1}{\partial x}\right) \tag{56}$$

in keeping with Eqs. (45–47).

As Eq. (56) shows, dispersion of spatial disturbances obeys the Laplace operator of the self-induced pressure. It remains homogeneous in both directions x and z and is not affected by nonlinearity of a set of governing equations. The type

$$\bar{P}_2 = \frac{i\omega k}{k^2+m^2}\bar{A}_2 - \frac{k^2}{2(k^2+m^2)}\bar{A}_1^2 \tag{57}$$

of the second-order relation differs from Eq. (40) by the last term on the right-hand side.

C. Third-Order Approximation

The form

$$(u_3, v_3, w_3, F_3, p_3, A_3) = [\bar{U}_3(y), \bar{V}_3(y), \bar{W}_3(y), \bar{F}_3, \bar{P}_3, \bar{A}_3] \exp\{3[\omega t + i(kx+mz)]\} \tag{58}$$

of the third-order travelling-wave-type solution is similar to Eqs. (38) and (44), but the argument $3[\omega t + i(kx+mz)]$ of the exponential factor becomes three times greater than the argument of the exponential factor in the linear solution. Accordingly, let us put

$$\bar{F}_3 = 3i(k\bar{U}_3 + m\bar{W}_3) \tag{59}$$

and introduce Eqs. (58) and (59) into Eq. (30c). As a result, we have

$$\begin{aligned}(\omega + iky)\bar{F}_3 + ik\bar{V}_3 - 3(k^2+m^2)\bar{P}_3 &= \frac{1}{3}\bar{Q}_3 \\ &= -\left\{ik\left[3\bar{U}_1(ik\bar{U}_2 + im\bar{W}_2) + \bar{V}_1\frac{d\bar{U}_2}{dy} + \bar{V}_2\frac{d\bar{U}_1}{dy}\right]\right. \\ &\quad \left. + im\left[3\bar{W}_1(ik\bar{U}_2 + im\bar{W}_2) + \bar{V}_1\frac{d\bar{W}_2}{dy} + \bar{V}_2\frac{d\bar{W}_1}{dy}\right]\right\}\end{aligned} \tag{60}$$

Taking into account that neither of the two functions $\bar{F}_1$ and $\bar{F}_2$ varies with y, and that their product

$$\bar{F}_1\bar{F}_2 = -2(k\bar{U}_1 + m\bar{W}_1)(k\bar{U}_2 + m\bar{W}_2)$$

the right-hand side of Eq. (60) simplifies to

$$\frac{1}{3}\bar{Q}_3 = -\frac{3}{2}\bar{F}_1\bar{F}_2 - \frac{1}{2}\bar{V}_1\frac{d\bar{F}_2}{dy} - \bar{V}_2\frac{d\bar{F}_1}{dy} = -\frac{3}{2}\bar{F}_1\bar{F}_2 \tag{61}$$

It follows that

$$(\omega + iky)\frac{d\bar{F}_3}{dy} = 0 \tag{62}$$

and the key property

$$F_3 = F_3(t, x, z) \tag{63}$$

of the auxiliary function F remains valid in the third approximation. The limit conditions (14) and (16) lead to the expression

$$\bar{F}_3 = 3ik\bar{A}_3 \tag{64}$$

of $\bar{F}_3$ in terms of $\bar{A}_3$ for any y. With $ik\bar{A}_1$, $2ik\bar{A}_2$, and $3ik\bar{A}_3$ substituted for $\bar{F}_1$, $\bar{F}_2$, and $\bar{F}_3$, respectively, Eq. (60) assumes the form

$$i\omega k\bar{A}_3 - (k^2+m^2)\bar{P}_3 = k^2\bar{A}_1\bar{A}_2 \tag{65}$$

where the boundary condition $\bar{V}_3 = 0$ at $y = 0$ is taken into account. None of the four members $\bar{A}_1$, $\bar{A}_2$, $\bar{A}_3$, and $\bar{P}_3$ here is a function of y. Since

$$\frac{\partial^2 A_3}{\partial t \partial x} = 9i\omega k\bar{A}_3 e^{3[\omega t + i(kx+mz)]}$$

the third-order evolution equation can be cast, in view of Eq. (65), as

$$\frac{\partial^2 A_3}{\partial t \partial x} + \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right) p_3 = -\frac{\partial^2 A_1 A_2}{\partial x^2} \tag{66}$$

Notice that dispersion of spatial disturbances governed by the Laplace operator of the self-induced pressure is retained homogeneously in x and z. Nonlinearity exerts no impact on the dispersion properties of travelling waves. However, the third-order relation

$$\bar{P}_3 = \frac{i\omega k}{k^2+m^2}\bar{A}_3 - \frac{k^2}{k^2+m^2}\bar{A}_1\bar{A}_2 \tag{67}$$

contains a correction term, depending on the first-order and the second-order components $\bar{A}_1$ and $\bar{A}_2$ of the displacement thicknesses.

V. General Consideration

The preceding analysis of the first two approximations was based on an expansion of the velocity and pressure fields into the power series of a small parameter, δ, and the independence of the auxiliary functions $\bar{F}_1$ and $\bar{F}_2$ on the lateral coordinate y. In fact, the condition that $\delta \ll 1$ was never used. The key property of $\bar{F}_1$ and $\bar{F}_2$ to be constant made it possible to prove that the third-order function $\bar{F}_3$ also does not vary with y. Let us set about to treat the higher-order approximations. A natural extension of the preceding line of reasoning is achieved by induction. Accordingly, suppose that

$$(u_n, v_n, w_n, F_n, p_n, A_n) = [\bar{U}_n(y), \bar{V}_n(y), \bar{W}_n(y), \bar{F}_n, \bar{P}_n, \bar{A}_n] \exp\{n[\omega t + i(kx+mz)]\} \tag{68}$$

but all the functions $\bar{F}_1, \bar{F}_2, \bar{F}_3, \ldots, \bar{F}_{n-1}$ entering the first-order ($n-1$-order) approximations are such that

$$\frac{d\bar{F}_1}{dy} = \frac{d\bar{F}_2}{dy} = \frac{d\bar{F}_3}{dy} = \ldots = \frac{d\bar{F}_{n-1}}{dy} = 0 \tag{69}$$

It must be proved that the function

$$\bar{F}_n = ni(k\bar{U}_n + m\bar{W}_n) = -\frac{d\bar{V}_n}{dy} \tag{70}$$

is also independent of y.

A differential consequence of equations of motion, which is analogous to Eqs. (49) and (60), reads

$$\begin{aligned}(\omega + iky)\bar{F}_n + ik\bar{V}_n - n(k^2+m^2)\bar{P}_n &= \frac{1}{n}\bar{Q}_n \\ &= -\left\{\frac{1}{2}nik[\bar{U}_1(ik\bar{U}_{n-1} + im\bar{W}_{n-1})\right. \\ &\quad + \bar{U}_2(ik\bar{U}_{n-2} + im\bar{W}_{n-2}) + \ldots] \\ &\quad + ik\left(\bar{V}_1\frac{d\bar{U}_{n-1}}{dy} + \bar{V}_2\frac{d\bar{U}_{n-2}}{dy} + \ldots\right) \\ &\quad + \frac{1}{2}nim[\bar{W}_1(ik\bar{U}_{n-1} + im\bar{W}_{n-1}) \\ &\quad + \bar{W}_2(ik\bar{U}_{n-2} + im\bar{W}_{n-2}) + \ldots] \\ &\quad \left. + im\left(\bar{V}_1\frac{d\bar{W}_{n-1}}{dy} + \bar{V}_2\frac{d\bar{W}_{n-2}}{dy} + \ldots\right)\right\}\end{aligned} \tag{71}$$

where all the series containing products of the type $\bar{\Phi}_j\bar{\Psi}_{n-j}$ terminate when the index j reaches a value $n-1$. On the assumption cast in Eq. (69), none of the first $n-1$ functions $\bar{F}_1, \bar{F}_2, \bar{F}_3, \ldots, \bar{F}_{n-1}$ depends on y; therefore, the right-hand side of Eq. (71) can be represented as

$$\frac{1}{n}\bar{Q}_n = -\frac{n}{2}\left(\frac{1}{n-1}\bar{F}_1\bar{F}_{n-1} + \frac{1}{2(n-2)}\bar{F}_2\bar{F}_{n-2} + \frac{1}{3(n-3)}\bar{F}_3\bar{F}_{n-3} + \ldots\right) - \left(\frac{1}{n-1}\bar{V}_1\frac{\mathrm{d}\bar{F}_{n-1}}{\mathrm{d}y} + \frac{1}{n-2}\bar{V}_2\frac{\mathrm{d}\bar{F}_{n-2}}{\mathrm{d}y} + \ldots\right)$$
$$= -\frac{n}{2}\left(\frac{1}{n-1}\bar{F}_1\bar{F}_{n-1} + \frac{1}{2(n-2)}\bar{F}_2\bar{F}_{n-2} + \frac{1}{3(n-3)}\bar{F}_3\bar{F}_{n-3} + \ldots\right) \tag{72}$$

with the series on the right-hand side terminating in the term $\bar{F}_j\bar{F}_{n-j}$, specified by $j = n-1$. When $n = 2$, Eq. (72) coincides with Eq. (50); for $n = 3$, Eq. (72) equals Eq. (61).

One can infer from Eqs. (71) and (72) that

$$(\omega + iky)\frac{\mathrm{d}\bar{F}_n}{\mathrm{d}y} = 0 \tag{73}$$

Thus, the key property

$$F_n = F_n(t, x, z) \tag{74}$$

of the auxiliary function F holds in the nth approximation if it holds in all the preceding $n-1$ approximations. Because Eq. (74) has been proven to be valid for $n = 2$ and $n = 3$, it is valid for an arbitrary n. From the limit conditions (14) and (16), we derive

$$\bar{F}_n = nik\bar{A}_n \tag{75}$$

and then

$$ni\omega k\bar{A}_n - n(k^2+m^2)\bar{P}_n = -\frac{1}{2}(nik)^2(\bar{A}_1\bar{A}_{n-1} + \bar{A}_2\bar{A}_{n-2} + \bar{A}_3\bar{A}_{n-3} + \ldots) \tag{76}$$

with the boundary condition $\bar{V}_n = 0$ at $y = 0$ being satisfied. Because the expression

$$\frac{\partial^2 A_n}{\partial t\partial x} = n^2 i\omega k\bar{A}_n e^{n[\omega t + i(kx+mz)]}$$

applies to an arbitrary term labeled by the subscript n, the product of $\exp\{n[\omega t + i(kx+mz)]\}$ by the sum

$$\frac{1}{2}(nik)^2(\bar{A}_1\bar{A}_{n-1} + \bar{A}_2\bar{A}_{n-2} + \bar{A}_3\bar{A}_{n-3} + \ldots)$$

provides the corresponding representation for the nth term in the expansion of the second derivative $1/2\partial^2A^2/\partial x^2$. Hence, the evolution equation governing A_n reads

$$\frac{\partial^2 A_n}{\partial t\partial x} + \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)p_n = -\frac{1}{2}\frac{\partial^2}{\partial x^2}(A_1A_{n-1} + A_2A_{n-2} + A_3A_{n-3} + \ldots) \tag{77}$$

In particular, the functions of the fourth approximation are related through

$$\frac{\partial^2 A_4}{\partial t\partial x} + \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)p_4 = -\frac{1}{2}\frac{\partial^2}{\partial x^2}(2A_1A_3 + A_2^2)$$

In order to obtain an explicit expression for the nonlinear term $\partial^2A^2/\partial x^2$ from its asymptotic expansion into a power series in δ, let us sum up Eqs. (36), (56), and (66),…, (77),…, multiplied by $\delta, \delta^2, \delta^3, \ldots, \delta^n, \ldots$, respectively. Each of the resulting series on the left-hand side consists of linear terms only, the first of which equals zero in view of Eq. (36). Therefore, summing all the terms on the left-hand side yields

$$\delta^2\left[\frac{\partial^2 A}{\partial t\partial x} + \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)p\right]$$

on the strength of Eqs. (26) and (27). When evaluating the right-hand side, it is advisable to revert to the spectral representations (38), (44), and (58),…, (68),…, because they offer the advantage of dealing with simple travelling-wave-type solutions. The series on the right-hand side converges to a limit that involves the displacement thickness $-A(t,x,z)$ expanded into components $A_1, A_2, \ldots, A_n, \ldots$, by means of Eq. (26). Closer examination shows that

$$\frac{1}{2}(ik)^2\{4\delta^2\bar{A}_1^2e^{2[\omega t+i(kx+mz)]} + 18\delta^3\bar{A}_1\bar{A}_2e^{3[\omega t+i(kx+mz)]} + 16\delta^4(2\bar{A}_1\bar{A}_3 + \bar{A}_2^2)e^{4[\omega t+i(kx+mz)]} + \ldots\} = \frac{1}{2}\delta^2\frac{\partial^2 A^2}{\partial x^2} \tag{78}$$

The factor δ^2 appears on both sides of the equation, so that it cancels. At this point, we may put $\delta = 1$ because this parameter drops out of the desired expression and the assumption that $\delta \ll 1$ has never been used. The dynamical system takes on the ultimate form

$$\frac{\partial^2 A}{\partial t\partial x} + \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)p = -\frac{1}{2}\frac{\partial^2 A^2}{\partial x^2} \tag{79}$$

indicated first by Ryzhov [21,22] using simple physical reasoning. Here, the interaction law (13) should be applied to connect the self-induced pressure p with the instantaneous displacement thickness $-A$. The important constraint $v = 0$ at $y = 0$ lies at the bottom of Eq. (79). A remarkable property is inherent in the preceding method of successive approximations. The travelling-wave-type solutions (38), (44), (58), and (68) underlie the first three steps, as well as the general proof. However, the expressions given in Eqs. (36), (56), and (66),…, (77) do not depend on the form of solutions and involve only the successive values $A_1, A_2, A_3, \ldots, A_n$ of the components of instantaneous displacement thickness. The final result in Eq. (79), defining the dynamical system, is also cast in terms of the total displacement thickness A, based on the sum of all the components $A_1, A_2, A_3, \ldots, A_n, \ldots$. Thus, the dynamical system derived has much broader applications than the travelling-type disturbances considered previously [23]. This property has deep repercussions when we set about to tackle oscillations provoked by an external agency.

VI. Forced Disturbances

The boundary condition (17) comes into operation, in place of Eq. (33), for a vibrator installed on a flat plate and shaped into $y = h(t,x,z)$. Let us supplement Eqs. (22–27) with the expansion

$$h = h' = \delta h_1 + \delta^2 h_2 + \delta^3 h_3 + \ldots + \delta^{n-1}h_{n-1} + \delta^n h_n + \ldots \tag{80}$$

and put

$$h_1 = \bar{H}_1\exp[\omega t + i(kx+mz)]\cdots h_n = \bar{H}_n\exp\{n[\omega t + i(kx+mz)]\} \tag{81}$$

similar to Eqs. (38), (44), and (58),…, (68). Then, the travelling-wave-type solution arises in each approximation, as the one induced by its respective forcing agency of the same type.

Under this condition, the sum of the two terms $u\partial h/\partial x$ and $w\partial h/\partial z$, entering the right-hand side of Eq. (17), is equal to

$$u\frac{\partial h}{\partial x}+w\frac{\partial h}{\partial z}=\delta^2\{ik\bar{H}_1[\bar{H}_1+\bar{U}_1(y)]+im\bar{H}_1\bar{W}_1(y)\}e^{2[\omega t+i(kx+mz)]}$$
$$+\delta^3\{2ik\bar{H}_2[\bar{H}_1+\bar{U}_1(y)]+ik\bar{H}_1[\bar{H}_2+\bar{U}_2(y)]$$
$$+2im\bar{H}_2\bar{W}_1(y)+im\bar{H}_1\bar{W}_2(y)\}e^{3[\omega t+i(kx+mz)]}+\dots$$
$$+\delta^n\{(n-1)ik\bar{H}_{n-1}[\bar{H}_1+\bar{U}_1(y)]+(n-2)ik\bar{H}_{n-2}[\bar{H}_2+\bar{U}_2(y)]$$
$$+(n-3)ik\bar{H}_{n-3}[\bar{H}_3+\bar{U}_3(y)]+(n-4)ik\bar{H}_{n-4}[\bar{H}_4+\bar{U}_4(y)]+\dots$$
$$+(n-1)im\bar{H}_{n-1}\bar{W}_1(y)+(n-2)im\bar{H}_{n-2}\bar{W}_2(y)$$
$$+(n-3)im\bar{H}_{n-3}\bar{W}_3(y)+\dots\}e^{n[\omega t+i(kx+mz)]}+\dots \tag{82}$$

By virtue of the definition of $\bar{F}_n$, given in Eq. (70), we derive, from Eq. (82), the expression

$$u\frac{\partial h}{\partial x}+w\frac{\partial h}{\partial z}=\delta^2\bar{H}_1(ik\bar{H}_1+\bar{F}_1)e^{2[\omega t+i(kx+mz)]}$$
$$+\delta^3\left[\bar{H}_2(2ik\bar{H}_1+2\bar{F}_1)+\bar{H}_1\left(ik\bar{H}_2+\frac{1}{2}\bar{F}_2\right)\right]e^{3[\omega t+i(kx+mz)]}$$
$$+\dots+\delta^n\left\{\bar{H}_{n-1}[(n-1)ik\bar{H}_1+(n-1)\bar{F}_1]\right.$$
$$\left.+\bar{H}_{n-2}\left[(n-2)ik\bar{H}_2+\frac{1}{2}(n-2)\bar{F}_2+\dots\right]\right\}e^{n[\omega t+i(kx+mz)]} \tag{83}$$

which does not include the velocity components $\bar{U}_j(y)$ and $\bar{W}_j(y)$, each taken separately. All the functions here are independent of the normal-to-wall distance y; therefore, Eq. (75) applies, reducing Eq. (83) to

$$u\frac{\partial h}{\partial x}+w\frac{\partial h}{\partial z}=\delta^2 ik\bar{H}_1(\bar{H}_1+\bar{A}_1)e^{2[\omega t+i(kx+mz)]}$$
$$+\delta^3[2ik\bar{H}_2(\bar{H}_1+\bar{A}_1)+ik\bar{H}_1(\bar{H}_2+\bar{A}_2)]e^{3[\omega t+i(kx+mz)]}+\dots$$
$$+\delta^n[(n-1)ik\bar{H}_{n-1}(\bar{H}_1+\bar{A}_1)+(n-2)ik\bar{H}_{n-2}(\bar{H}_2+\bar{A}_2)+\dots]$$
$$\times e^{n[\omega t+i(kx+mz)]}+\dots \tag{84}$$

The coefficients of the exponential terms in Eq. (84) pertain equally to a vibrator inducing a two-dimensional disturbance pattern, for which

$$u=y+A,\qquad \frac{\partial v}{\partial x}=-\left(y\frac{\partial^2 A}{\partial x^2}+\frac{\partial^2 A}{\partial t\partial x}+\frac{1}{2}\frac{\partial^2 A^2}{\partial x^2}+\frac{\partial^2 p}{\partial x^2}\right) \tag{85}$$

is an exact solution of the set of governing asymptotic Eqs. (10) and (11), with Eq. (12) identically vanishing to zero, insofar as $w=\partial/\partial z=0$ (see Bogdanova-Ryzhova and Ryzhov [28] and Ryzhov and Bogdanova-Ryzhova [17]). The boundary condition (17), which simplifies in this case to

$$v=\frac{\partial h}{\partial t}+(y+A)\frac{\partial h}{\partial x}\quad \text{at } y=h(t,x) \tag{86}$$

can be exploited generally, because both Eqs. (70) and (75), specifying the normal-to-wall component of velocity, hold independent of wave-field dimensionality.

With these results in hand, we can set about to develop a dynamical system covering the emission of short-scaled large-sized disturbances by an external agency. An important point needs to be made. The differential consequence (71) of equations of motion can be cast in the form

$$ikn\bar{V}_n=-\omega ikn^2\bar{A}_n-y(ikn)^2\bar{A}_n-n^2(ik+im)\bar{P}_n$$
$$-\frac{1}{2}(ikn)^2(\bar{A}_1\bar{A}_{n-1}+\bar{A}_2\bar{A}_{n-2}+\dots) \tag{87}$$

at any distance y from the body, provided that no boundary condition is imposed on the wall. As is readily seen, the limit condition (15),

$$\frac{\partial v}{\partial x}=-\left[y\frac{\partial^2 A}{\partial x^2}+\frac{\partial^2 A}{\partial t\partial x}+\frac{1}{2}\frac{\partial^2 A^2}{\partial x^2}+\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)p\right] \tag{88}$$

leads to the same relation (87) at the outer edge of the adjustment sublayer. This means that the limit condition as $y\to\infty$ remains valid throughout the entire sublayer. From Eq. (86), we have

$$\frac{\partial v}{\partial x}=\frac{\partial}{\partial x}\left[\frac{\partial h}{\partial t}+(h+A)\frac{\partial h}{\partial x}\right]\quad \text{at } y=h(t,x,z) \tag{89}$$

When evaluating the derivative $\partial v/\partial x$ on the vibrator surface, its location $y=h(t,x,z)$ should be taken into account. Then, the expression

$$\frac{\partial v}{\partial x}=-\left[h\frac{\partial^2 A}{\partial x^2}+\frac{\partial h}{\partial x}\frac{\partial A}{\partial x}+\frac{\partial^2 A}{\partial t\partial x}+\frac{1}{2}\frac{\partial^2 A^2}{\partial x^2}+\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)p\right] \tag{90}$$

comes in place of Eq. (88), in which y is an independent variable. Equating Eqs. (89) and (90), we have

$$\frac{\partial^2 h}{\partial t\partial x}+A\frac{\partial^2 h}{\partial x^2}+h\frac{\partial^2 h}{\partial x^2}+2\frac{\partial A}{\partial x}\frac{\partial h}{\partial x}+\left(\frac{\partial h}{\partial x}\right)^2+h\frac{\partial^2 A}{\partial x^2}$$
$$+\frac{\partial^2 A}{\partial t\partial x}+\frac{1}{2}\frac{\partial^2 A^2}{\partial x^2}+\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)p=0 \tag{91}$$

A similar equation, with the term $\partial^2 p/\partial z^2$ omitted, controls the radiation of two-dimensional signals. Taking advantage of this analogy, we derive the dynamical system

$$\frac{\partial^2 A_w}{\partial t\partial x}+\frac{1}{2}\frac{\partial^2 A_w^2}{\partial x^2}+\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)p=0 \tag{92}$$

sought by introducing a generalized displacement thickness (see Bogdanova-Ryzhova and Ryzhov [28] and Ryzhov and Bogdanova-Ryzhova [17]):

$$A_w=A+h \tag{93}$$

The dependence on the vibrator shape stems from the interaction law, which relates the self-induced pressure to the instantaneous displacement thickness and is assumed to be linear. Inserting Eq. (13) into Eq. (92) yields

$$\frac{\partial^2 A_w}{\partial t\partial x}+\frac{1}{2}\frac{\partial^2 A_w^2}{\partial x^2}+\mathcal{L}\left[\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)A_w\right]=\mathcal{L}\left[\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)h\right] \tag{94}$$

Like Eq. (79), the dynamical system under discussion has a broad spectrum of applications apart from those arising in connection with travelling-type disturbances.

VII. Birth of Erratic Signals and Large Coherent Structures

Let the operator $\mathcal{L}(A)$, in Eq. (13), be

$$\mathcal{L}=-\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}\xi\int_{-\infty}^{\infty}\frac{\partial^2 A/\partial\xi^2}{[(x-\xi)^2+(z-\zeta)^2]^{1/2}}\,\mathrm{d}\zeta+\Delta_0\frac{\partial^2 A}{\partial x^2} \tag{95}$$

where the first term on the right-hand side is brought about by the outer potential flow/adjustment sublayer interaction (see Stewartson [26] and Smith [27]); the second term, proportional to Δ_0, originates from allowance made for the curvature of stream surfaces, which supports centrifugal forces in balance with the transverse component of the pressure gradient [17]. For a wave system spreading in the Blasius boundary layer, Eq. (95) reduces to the two-dimensional extension

$$\mathcal{L} = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\xi \int_{-\infty}^{\infty} \frac{\partial^2 A/\partial \xi^2}{[(x-\xi)^2 + (z-\zeta)^2]^{1/2}}\, \mathrm{d}\zeta \tag{96}$$

of the BDA dispersion, if the stream-surface curvature is not taken into account ($\Delta_0 = 0$).

For a near-wall jet propagating along a flat plate submerged in a quiescent fluid, the integral term on the right-hand side of Eq. (95) identically vanishes to zero, whereas Δ_0 can be equated with one giving rise to the KDV dispersion law:

$$\mathcal{L} = -\frac{\partial^2 A}{\partial x^2} \tag{97}$$

As a result, Eq. (79) turns into

$$\frac{\partial A}{\partial t} + A\frac{\partial A}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)A \tag{98}$$

Consider, briefly, the travelling-wave-type solution

$$A = \tilde{A}(x_{1,}z), \qquad x_1 = x - ct, \qquad c = \mathrm{const}$$

of the generalized two-dimensional KDV equation, which comes from

$$\left(\frac{1}{2}\tilde{A} - c\right)\tilde{A} = \frac{\partial^2 \tilde{A}}{\partial x_1^2} + \frac{\partial^2 \tilde{A}}{\partial z^2}$$

and has the disturbance amplitude that does not depend on time. A remarkable property of the solution is associated with the invariance of the Laplace operator on the right-hand side, with respect to rotation:

$$\xi_1 = x_1\cos\varphi - z\sin\varphi, \qquad \eta_1 = x_1\sin\varphi + z\cos\varphi$$

of a frame of reference. A solitary wave travelling in the direction of the oncoming stream can be used to obtain a coherent structure of the same planform that moves in any direction. So, there exists a continuum of equally shaped nonlinear disturbances inclined at an arbitrary angle with respect to the base flow. The simple axisymmetric solution

$$\tilde{A} = \tilde{A}(r_1)$$

satisfies the ordinary differential equation

$$\left(\frac{1}{2}\tilde{A} - c\right)\tilde{A} = \frac{1}{r_1}\frac{\mathrm{d}}{\mathrm{d}r_1}\left(r_1\frac{\mathrm{d}\tilde{A}}{\mathrm{d}r_1}\right)$$

where $r_1 = \sqrt{x_1^2 + z^2}$ denotes the radial distance in the moving frame of reference x_1, z. The requirement $\tilde{A} \to 0$ as $r_1 \to \infty$ defines solitons that are nothing but nonlinear eigenfunctions, mentioned in relation to the general boundary condition (17). The phase velocity c and the minimum of $\tilde{A}$ cannot be chosen arbitrarily: with the reference value $c = -1$, we obtain $\min\tilde{A}(-1) = -4.776$ for the disturbance moving in the x direction. In the general case, the phase velocity becomes $c = \min A(c)/4.776$.

The alternative form

$$\frac{\partial}{\partial x}\left(\frac{\partial A}{\partial t} + A\frac{\partial A}{\partial x} - \frac{\partial^3 A}{\partial x^3}\right) = \frac{\partial^2 A}{\partial z^2} \tag{99}$$

of the two-dimensional generalization of the KDV equation has been put forward by Kadomtsev and Petviashvili [18]. The KP equation describes the long-wave motion if it is weakly nonlinear, weakly dispersive, and weakly two-dimensional, with all three effects being of the same order. Both Eqs. (98) and (99) converge to the classical KDV equation, provided that $\partial/\partial z = 0$.

It is worth noting that Eq. (94), with $\mathcal{L}$ prescribed by Eq. (95), admits of the steady-state solution $A_w = A_w(x, z)$, satisfying the equation

$$\frac{\partial^2 A_w^2}{\partial x^2} + 2\mathcal{L}\left[\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)A_w\right] = 2\mathcal{L}\left[\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)h\right] \tag{100}$$

However, the steady-state solutions seem to be inherently unstable. To demonstrate this, we consider a local hump:

$$h = \sigma\cos^2\frac{\pi x}{2b}, \qquad -b < x < b \tag{101}$$

of the constant height σ, placed on an otherwise flat surface $y = 0$, and stretching from $z = -\infty$ to $z = \infty$. The hump is assumed to gradually emerge on the plate on an initial interval $0 < t < 1$, and then to preserve its shape over a long period $1 < t < 900$ of time. Taking advantage of an affine transformation, the dynamical system becomes

$$\frac{\partial A_w}{\partial t} + A_w\frac{\partial A_w}{\partial x} - \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\partial^2 A_w/\partial\xi^2}{\xi - x}\,\mathrm{d}\xi - \Delta_0\frac{\partial^3 A_w}{\partial x^3} = f(t.x) \tag{102a}$$

$$f = Q\left(\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\partial^2 h/\partial\xi^2}{\xi - x}\,\mathrm{d}\xi - \Delta_0\frac{\partial^3 h}{\partial x^3}\right) \tag{102b}$$

It involves dispersion terms typical of both BDA and KDV equations, each taken separately. A similarity parameter Q is a function of σ and b. A compound BDA–KDV system without forcing ($f = 0$) appeared in the work by Benjamin [14] on solitary waves in a two-fluid system. Independently, a similar equation was advanced as applied to completely different environmental conditions and spectral range, in an invited lecture delivered by Ryzhov [15] at the First European Fluid Mechanics Conference. The focus in this lecture was on nonlinear stability of the boundary-layer flow. Transition to turbulence is a new aspect, designed to shed light on the interplay between irregular and well-organized motion.

The pseudospectral scheme that forms the basis for computed results, described next, was first developed in Burggraf and Duck [29] and Duck [30], in which many technical details were carefully examined and accuracy was quantified depending on different conditions. The incorporation of the fast Fourier transform algorithm allows spectral computations to be performed with sufficient speed. Chuang and Conlisk [31] argue that this method is even less time-consuming, for approximately half the number of grid points are necessary to bring the results to within a few percentage points of the corresponding values, calculated by means of finite difference schemes. A comprehensive comparison of the two numerical procedures is available in Bodonyi et al. [32]. Since its invention, the scheme under discussion has been applied to the investigation of many types of highly oscillatory motions: turbine- and compressor-blade flows are typical recent examples (see Ryzhov [33]).

The pseudospectral scheme seems to be especially suitable for resolving a disturbance field, in which the large-sized solitarylike waves spring up and propagate against the background of lower-amplitude, erratically distributed signals. The number of Fourier modes used in computing Figs. 1 and 2 is 2^{15}. This amount of Fourier modes proved to be sufficient to handle the flow reversal, imbedded in the large-sized coherent structures close to a solid surface (not shown). The flow reversal briefly mentioned by Ryzhov [33], in connection with a few central cycles of wave packets vigorously building up into narrow peaks, presents a challenge to the finite difference computation that is missing from the spectral approach.

Figure 1 presents the general view of a typical wave system, and three coherent solitarylike disturbances are scaled up in Fig. 2. The "foam" of erratic oscillations is seen to sweep downstream of the obstacle. Solitarylike waves emerge inside the foam from irregular signals. Emerging coherent structures survive the random buffeting by the surrounding fluid and continue to build up over long distances. A large coherent disturbance becomes completely shaped into a solitary wave as early as $t = 100$. Then, it breaks away from continuously emitted pulsations and rapidly moves as a precursor of

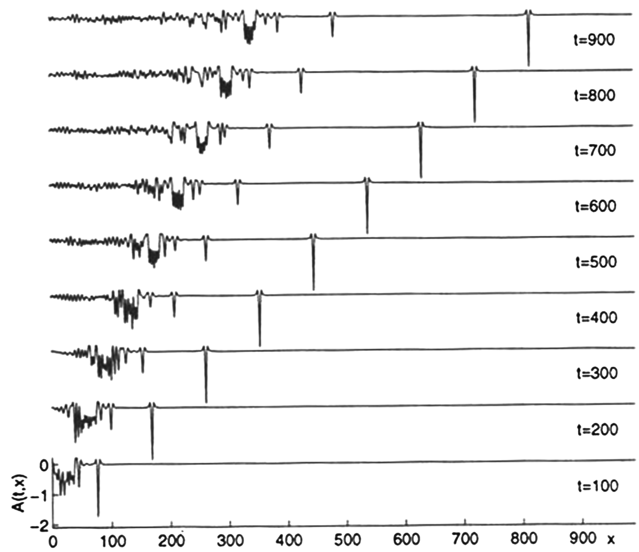

Fig. 1 Birth of large-sized solitarylike waves, from small-sized erratic signals.

the oscillatory motion. The next solitary wave starts building up in the depression zone filled by erratic pulsations, and at some time between $t = 200$ and $t = 300$ it separates from them, turning into the second precursor. By the time $t = 900$, the third coherent disturbance emerges in front of the oscillatory motion. The existence of the solitary waves can be predicted rigorously, but the place of their birth does not obey any apparent law.

The shape of the solitary wave in Fig. 2c differs from those shown in Figs. 2a and 2b. The distinction between them also correlates well with rigorous results, indicative of several families of nonlinear eigenfunctions intrinsic to the compound BDA–KDV system. Upon breaking away from the low-amplitude erratic oscillatory motion, all three solitary waves in Figs. 1 and 2 tend to arrange themselves into one of the eigenfunctions intrinsic to the compound BDA–KDV system. However, as distinct from a simple example considered previously, starting from the generalized two-dimensional KDV equation, these three coherent structures grow out of the collisions of small-sized pulsations, rather than moving against a zero background. Thus, the flow–obstacle interaction gives rise both to the irregular pulsations and to the well-developed solitarylike disturbances. The inclusion of streamline curvature effects in the interaction law, intrinsic to the compound BDA–KDV system, is of fundamental importance, because neither of the parent BDA and KDV evolution equations contain solitons travelling in front of low-amplitude oscillations (see Ryzhov and Bogdanova-Ryzhova [17]).

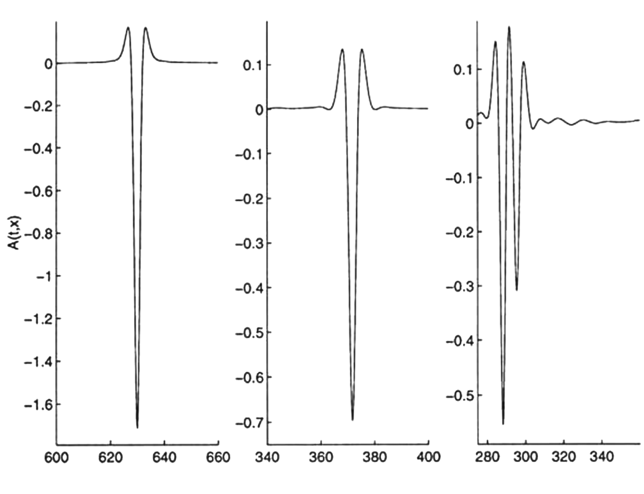

Fig. 2 Scaled-up view of solitarylike coherent structures, shown in Fig. 1 to arise from erratic signals.

VIII. Conclusions

Let the results achieved be summed up and a few concluding remarks made. A new two-dimensional dynamical system is shown to control the three-dimensional disturbance pattern in a shear flow transitioning to turbulence under the action of external forcing. The general formulation is given in terms of the self-induced pressure and the instantaneous displacement thickness. An interaction law is required to close the definition of the problem, by relating both these quantities to each other. The type of the interaction law depends on the surroundings.

An asymptotic power series in a small parameter underlies the derivation of a quadratic term entering the dynamical system. As usual, an infinite set of linear equations is inherent in the method of successive approximations. Travelling-wave-type functions are introduced, to solve the linear equations. The spectral solutions are summed up, to rigorously determine the quadratic term in the dynamical system, regardless of the form of the interaction law. An analogous technique proves to be successful, in dealing with the expression for an external perturbing agency. As a consequence, the final definition of the dynamical system cast in Eq. (94) does not depend at all on the travelling-wave-type functions applied when solving the linear equations of successive approximations.

It should be stated that the rise of stochastic pulsations is the much-studied feature of transition. Stochastic pulsations are characteristic of fully developed turbulence in open flows, as well as in channels and pipes. On the other hand, strong nonlinear coherent structures also simultaneously originate in the velocity field as an integral part of transition (see Cantwell [24] and Kachanov [25]). Therefore, a dynamical system intended to describe transition should be capable of shedding light on the interplay between erratic and well-organized motions. The simplest one-dimensional case, specified by $w = \partial/\partial z = 0$ and designed to examine Eq. (94) for this capability, testifies that, in accord with experimental evidence, erratic low-amplitude signals and much stronger coherent structures come hand in hand, being inextricably entwined. Certainly, it is necessary to keep in mind that three-dimensional stochastic oscillations feature turbulence, so the computation assuming $w = \partial/\partial z = 0$ cannot be considered as its comprehensive model. Owing to the two-dimensional character of disturbances, exhibited in Figs. 1 and 2, they do not represent fully developed turbulent fields. Rather, these figures are intended to demonstrate that the dynamical system derived is capable of predicting the emission of erratic signals by a steady roughness, the collisions of which give rise to large-sized solitarylike waves. Qualitatively, the process is very much akin to that known from the wind-tunnel tests. However, even ample evidence reported in Cantwell [24] and Kachanov [25], as well as in numerous papers cited therein, does not provide a solid background to be compared with. The present study calls for additional thorough observations. As for the theory of solitarylike waves governing three-dimensional flows, preliminary results relating to the compound BDA–KDV system point to several families of pertinent solutions.

The inclusion of streamline curvature effects in the interaction law (95) seems to be of fundamental importance to the coexistence of weak, randomly distributed pulsations and strong solitarylike disturbances, owing to the fact that the BDA and KDV dispersions relate to different spectral ranges.

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